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THE FORMAL LINEARIZATION METHOD TO MULTISOLITON SOLUTIONS FOR THREE MODEL EQUATIONS OF SHALLOW WATER WAVES

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Abstract. In this paper, the formal linearization method is used to construct multisoliton solutions for three model of shallow water waves equations. The three models are completely integrable. The formal linearization method is an efficient method for obtaining exact multisoliton solutions of nonlinear partial differential equations. The method can be applied to nonintegrable equations as well as to integrable ones.

1. Introduction

Wazwaz [1] applied the Hirota's bilinear method for obtaining multisoliton solutions of three model of shallow water waves equations in following forms:
The first shallow water waves equation

$$u_{xxt} + 3uu_t + 3u_x \int^x u_t \, dx - u_x - u_t = 0. \tag{1.1}$$

The second shallow water waves equation

$$u_{xxt} + 3uu_t + 3u_x \int^x u_t \, dx - u_{xxx} - 6uu_x - u_t = 0. \tag{1.2}$$

The third shallow water waves equation

$$u_{xxt} + 3uu_t + 3u_x \int^x u_t \, dx - u_x - u_{xxx} - 6uu_x - u_t = 0, \tag{1.3}$$

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where $u = u(t, x)$. The aim of this paper is to find new multisoliton solutions of three model of shallow water waves equations by using the formal linearization method [2,3].

2. Formal linearization method

Let us consider equations of the following form

$$\hat{L}(D_t, D_x)u(t,x) = N[u], \tag{2.1}$$

where

$$\hat{L}(D_t, D_x) = \sum_{k=0}^{K}\sum_{m=0}^{M} l_{km} D_t^k D_x^m \tag{2.2}$$

is a linear differential operator with constant coefficients and

$$N[u] = N(u, u_1, u_2, \ldots, u_p),\ u_p = \frac{\partial^{p_1+p_2}u}{\partial t^{p_1}\partial x^{p_2}},\ p = (p_1, p_2)$$

is an arbitrary analytic function of u and of its derivatives up to some finite order p. We suppose that Eq.(4) possesses the constant solution. Without loss of generality we assume that

$$N[0] = 0,\ \frac{\partial N[0]}{\partial u} = 0,\ \frac{\partial N[0]}{\partial u_1} = 0, \ldots,\ \frac{\partial N[0]}{\partial u_p} = 0.$$

We consider Eq.(4) in connection with the equation linearized near a zero solution:

$$\hat{L}(D_t, D_x)w(t,x) = 0 \tag{2.3}$$

Let L be the vector space of solutions of Eq.(6) and $P^N \subset L$ be the N-dimensional subspace with the basis

$$w_i = W_i exp(\alpha_i \xi_i),\ \xi_i = x - s_i t,\ i = 1, .., N.$$

Here s_i and W_i are some constants.The constants $\alpha_i = \alpha_i(s_i)$ are assumed to satisfy the dispersion relation

$$\hat{L}(-\alpha_i s_i, \alpha_i) = 0.$$

The subspace$P^N = \{\sum_{i=1}^{N} C_i w_i | C_i = const\}$ is specified by the system of N linear ordinary differential equations

$$\frac{dw_i}{d\xi_i} = \alpha_i w_i,\ i = 1, .., N.$$

We use the following notation:

$$w_{(N)}^{\delta} = w_1^{\delta_1} w_2^{\delta_2} \ldots w_N^{\delta_N}$$

$$\delta = (\delta_1, \delta_2, \ldots, \delta_N)$$

$$|\delta| = \sum_{i=1}^{N} \delta_i.$$

It is obvious that the monomials $w_{(N)}^{\delta}$ are the eigenfunctions of the operator (5):

$$\hat{L}(D_t, D_x) w_{(N)}^{\delta} = \lambda_\delta w_{(N)}^{\delta}$$

with the eigenvalues

$$\lambda_\delta = \sum_{k=0}^{K} \sum_{m=0}^{M} l_{km} (-\sum_{i=1}^{N} \alpha_i s_i \delta_i)^k (\sum_{i=1}^{N} \alpha_i \delta_i)^m.$$

THEOREM 2.1. *If $\lambda_\delta \neq 0$ for every multiindex δ with positive integer components $\delta_i \in Z_+, i = 1, ..., N$, satisfying the condition $|\delta| \neq 0, 1$, then Eq.(4) possesses solutions connected with solutions form P^N by the formal transformation*

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2, \ldots, w_N), \tag{2.4}$$

where

$$\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(N)}^{\delta} \tag{2.5}$$

are homogeneous polynomials of degree n in the variables w_i .This transformation is unique(for the first term $\phi_1 \in P^N$ fixed).

REMARK 2.2. Here ε is the grading parameter ,finally we can put $\varepsilon = 1$.
The proof of the theorem is constructive. Substituting (7) into (4), expanding $N[u]$ into the power series in ε ,and then collecting equal powers of ε, we obtain the determining equations for the functions ϕ_n and show that if $\lambda_\delta \neq 0$,then these equations possess the solution (8) with the coefficients $(A_n)_\delta$ uniquely determined through the coefficients $(A_1)_\delta$ by the recursion relation.Thus, the theorem gives us the method for constructing particular solutions of *Eq.*(4).

3. The first model equation for shallow water waves

In this section, we apply the formal linearization to the first shallow water waves equation:

$$u_{xxt} + 3uu_t + 3u_x \int^x u_t \, dx - u_x - u_t = 0. \tag{3.1}$$

Using the potential

$$u = v_x, \tag{3.2}$$

Eq.(9) becomes

$$v_{xxxt} + 3v_x v_{xt} + 3v_{xx} v_t - v_{xx} - v_{xt} = 0. \tag{3.3}$$

Integrating (11) with respect to x and neglecting the constant of integration we obtain

$$v_{xxt} + 3v_x v_t - v_x - v_t = 0. \tag{3.4}$$

Thus, Eq.(12) can write in the form

$$\hat{L}(D_t, D_x)v(t,x) = -3v_x v_t, \tag{3.5}$$

$$\hat{L}(D_t, D_x) = D_x^2 D_t - D_x - D_t.$$

For simplicity we look for a solution of Eq.(13) in the form

$$v(t,x) = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2), \tag{3.6}$$

where

$$w_i = W_i exp[\sqrt{\frac{s_i - 1}{s_i}}(x - s_i t)], \ i = 1, 2,$$

is the basis of the subspace $P^2 \subset L$(let s_i and W_i be some real constants). Substituting (14) into Eq.(13) and collecting equal powers of ε we obtain the determining equations for the functions ϕ_n as follows

$$\hat{L}\phi_1 = 0,$$

$$\hat{L}\phi_n = -3\sum_{k=1}^{n-1} D_x \phi_k D_t \phi_{n-k}, \ n \geq 2. \tag{3.7}$$

These equations possess the solution $\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(2)}^\delta$, $\delta = (\delta_1, \delta_2)$, which can be rewritten in this case in the following form

$$\phi_n = \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k} \ (\phi_1 \in P^2), \tag{3.8}$$

the coefficients A_k^n can be found through A_0^1 and A_1^1 (we can assume that either $A_0^1 = A_1^1 = 1$ or $A_0^1 = 0, A_1^1 = 1$)by the recursion relation:
If $n \geq 2,\ 0 \leq k \leq n$ then

$$\begin{aligned} A_k^n &= \frac{3}{\lambda_{(k,n-k)}} \sum_{l=1}^{n-1} \sum_{m=0}^{n-l} [\ \sqrt{\frac{s_1-1}{s_1}}(k-m) + \sqrt{\frac{s_2-1}{s_2}}(l-k+m)] \\ &[\quad s_1\sqrt{\frac{s_1-1}{s_1}}\ m + s_2\sqrt{\frac{s_2-1}{s_2}}(n-l-m)] A_{k-m}^l A_m^{n-l}, \end{aligned}$$

if $k < 0$ or $k > n$ then $A_k^n = 0$,

$$\begin{aligned} \lambda_{(k,n-k)} &= (s_1-1)\sqrt{\frac{s_1-1}{s_1}} k(1-k^2) \\ &+ (s_2-1)\sqrt{\frac{s_2-1}{s_2}}(n-k)(1-(n-k)^2) \\ &- \frac{(s_1-1)(s_2+2s_1)}{s_1}\sqrt{\frac{s_2-1}{s_2}} k^2(n-k) \\ &- \frac{(s_2-1)(s_1+2s_2)}{s_2}\sqrt{\frac{s_1-1}{s_1}} k(n-k)^2. \end{aligned}$$

If $s_1 < 0,\ s_1 > 1$ and $s_2 < 0,\ s_2 > 1$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $k, n \in Z_+,\ n \geq 2,\ 0 \leq k \leq n$.
Thus, By (10), (14) we obtain a 2-soliton solution of the first shallow water waves equation in the form:

$$u(t,x) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} (\sqrt{\frac{s_1-1}{s_1}} k + \sqrt{\frac{s_2-1}{s_2}}(n-k)) A_k^n w_1^k w_2^{n-k}.$$

Remark 3.1. If $A_0^1 = 0$, then $\phi_1 \in P^1$ and we get form (14) the expansion for a 1-soliton solution. For obtaining the N-soliton solution, we must take $\phi_1 \in P^N$.

If $A_0^1 = 0$, then by (14) we obtain

$$\begin{aligned} v &= \sum_{n=1}^{\infty} (-\frac{1}{2})^{n-1} (\sqrt{\frac{s_1-1}{s_1}})^{n-1} (\varepsilon w_1)^n \\ &= \frac{\varepsilon w_1}{1 + \frac{1}{2}\sqrt{\frac{s_1}{s_1-1}}\varepsilon w_1} = \sqrt{\frac{s_1-1}{s_1}} \frac{2w}{1+w}, \end{aligned} \tag{3.9}$$

where $w = \frac{\varepsilon}{2}\sqrt{\frac{s_1}{s_1-1}}w_1$.
In (t,x)-variables we have:

$$v(t,x) = \sqrt{\frac{s_1-1}{s_1}}(1 + tanh[\frac{1}{2}\sqrt{\frac{s_1-1}{s_1}}(x - s_1 t)]),$$

where we assumed $W_1 = \frac{2}{\varepsilon}\sqrt{\frac{s_1-1}{s_1}}$.
Therefore, By (10) we obtain exact soliton solution of the first shallow water waves equation in the form

$$u(t,x) = \frac{s-1}{2s}sech^2[\frac{1}{2}\sqrt{\frac{s-1}{s}}(x - st)].$$

4. The second model equation for shallow water waves

In this section, we apply the formal linearization to the second shallow water waves equation:

$$u_{xxt} + 3uu_t + 3u_x \int^x u_t\, dx - u_{xxx} - 6uu_x - u_t = 0. \tag{4.1}$$

Using of (10), Eq.(18) becomes

$$v_{xxxt} + 3v_x v_{xt} + 3v_{xx}v_t - v_{xxxx} - 6v_x v_{xx} - v_{xt} = 0. \tag{4.2}$$

Integrating (19) with respect to x and neglecting the constant of integration we obtain

$$v_{xxt} + 3v_x v_t - v_{xxx} - 3(v_x)^2 - v_t = 0. \tag{4.3}$$

Thus, Eq.(20) can write in the form

$$\hat{L}(D_t, D_x)v(t,x) = 3(v_x)^2 - 3v_x v_t, \tag{4.4}$$
$$\hat{L}(D_t, D_x) = D_x^2 D_t - D_x^3 - D_t.$$

The equation linearized near a zero solution has the form $\hat{L}w = 0$, and the space of its solutions contains the subspace P^2 with the basis

$$w_i = W_i exp[\sqrt{\frac{s_i}{s_i+1}}(x - s_i t)],\ i = 1,2.$$

We look for solutions of (21) in the form (14) and obtain the determining equation as follows:

$$\hat{L}\phi_1 = 0,$$
$$\hat{L}\phi_n = 3(\sum_{k=1}^{n-1} D_x\phi_k D_x\phi_{n-k} - \sum_{k=1}^{n-1} D_x\phi_k D_t\phi_{n-k}),\ n \geq 2. \tag{4.5}$$

These equations possess the solution

$$\phi_n = \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k} \ (\phi_1 \in P^2), \tag{4.6}$$

the coefficients A_k^n can be found through A_0^1 and A_1^1 (we can assume that either $A_0^1 = A_1^1 = 1$ or $A_0^1 = 0, A_1^1 = 1$) by the recursion relation: If $n \geq 2,\ 0 \leq k \leq n$ then

$$\begin{aligned} A_k^n &= \frac{3}{\lambda_{(k,n-k)}}\{\sum_{l=1}^{n-1}\sum_{m=0}^{n-l}[\sqrt{\frac{s_1}{s_1+1}}(k-m)+\sqrt{\frac{s_2}{s_2+1}}(l-k+m)] \\ &\quad [\sqrt{\frac{s_1}{s_1+1}}\,m+\sqrt{\frac{s_2}{s_2+1}}(n-l-m)]A_{k-m}^l A_m^{n-l} \\ &+ \sum_{l=1}^{n-1}\sum_{m=0}^{n-l}[\sqrt{\frac{s_1}{s_1+1}}(k-m)+\sqrt{\frac{s_2}{s_2+1}}(l-k+m)] \\ &\quad [s_1\sqrt{\frac{s_1}{s_1+1}}\,m+s_2\sqrt{\frac{s_2}{s_2+1}}(n-l-m)]A_{k-m}^l A_m^{n-l}\}, \end{aligned}$$

if $k < 0$ or $k > n$ then $A_k^n = 0$,

$$\begin{aligned} \lambda_{(k,n-k)} &= s_1\sqrt{\frac{s_1}{s_1+1}}k(1-k^2) \\ &+ s_2\sqrt{\frac{s_2}{s_2+1}}(n-k)(1-(n-k)^2) \\ &- \frac{2s_1^2+s_1s_2+3s_1}{s_1+1}\sqrt{\frac{s_2}{s_2+1}}k^2(n-k) \\ &- \frac{2s_2^2+s_1s_2+3s_2}{s_2+1}\sqrt{\frac{s_1}{s_1+1}}k(n-k)^2. \end{aligned}$$

If $s_1 < -1$, $s_1 > 0$ and $s_2 < -1$, $s_2 > 0$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $k, n \in Z_+$, $n \geq 2$, $0 \leq k \leq n$.
Thus, we obtain a 2-soliton solution of the second shallow water waves equation in the form:

$$u(t,x) = \sum_{n=1}^{\infty}\sum_{k=0}^{n}(\sqrt{\frac{s_1}{s_1+1}}k+\sqrt{\frac{s_2}{s_2+1}}(n-k))A_k^n w_1^k w_2^{n-k}.$$

If $A_0^1 = 0$, then by (14) we obtain

$$v = \sum_{n=1}^{\infty}(-\frac{1}{2})^{n-1}(\sqrt{\frac{s_1+1}{s_1}})^{n-1}(\varepsilon w_1)^n = \frac{\varepsilon w_1}{1+\frac{1}{2}\sqrt{\frac{s_1+1}{s_1}}\varepsilon w_1} = \sqrt{\frac{s_1}{s_1+1}}\frac{2w}{1+w}, \tag{4.7}$$

where $w = \frac{\varepsilon}{2}\sqrt{\frac{s_1+1}{s_1}}w_1$.
In (t, x)-variables we have:

$$v(t,x) = \sqrt{\frac{s_1}{s_1+1}}(1 + tanh[\frac{1}{2}\sqrt{\frac{s_1}{s_1+1}}(x - s_1 t)]),$$

where we assumed $W_1 = \frac{2}{\varepsilon}\sqrt{\frac{s_1}{s_1+1}}$.
Therefore, By (10) we obtain exact soliton solution of the second shallow water waves equation in the form

$$u(t,x) = \frac{s}{2(s+1)}sech^2[\frac{1}{2}\sqrt{\frac{s}{s+1}}(x - st)].$$

5. The third model equation for shallow water waves

We finally apply the formal linearization to the third shallow water waves equation:

$$u_{xxt} + 3uu_t + 3u_x\int^x u_t\,dx - u_x - u_{xxx} - 6uu_x - u_t = 0, \tag{5.1}$$

Using of (10), Eq.(25) becomes

$$v_{xxxt} + 3v_x v_{xt} + 3v_{xx}v_t - v_{xx} - v_{xxxx} - 6v_x v_{xx} - v_{xt} = 0. \tag{5.2}$$

Integrating (26) with respect to x and neglecting the constant of integration we obtain

$$v_{xxt} + 3v_x v_t - v_x - v_{xxx} - 3(v_x)^2 - v_t = 0. \tag{5.3}$$

Thus, Eq.(27) can write in the form

$$\hat{L}(D_t, D_x)v(t,x) = 3(v_x)^2 - 3v_x v_t, \qquad \hat{L}(D_t, D_x) = D_x^2 D_t - D_x - D_x^3 - D_t. \tag{5.4}$$

In this case, the subspace P^2 is generated by the functions

$$w_i = W_i exp[\sqrt{\frac{s_i-1}{s_i+1}}(x - s_i t)],\ i = 1,2.$$

Our procedure gives

$$v(t,x) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k}, \tag{5.5}$$

$$\begin{aligned}
A_k^n \;=\;& \frac{3}{\lambda_{(k,n-k)}}\{\sum_{l=1}^{n-1}\sum_{m=0}^{n-l}[\;\sqrt{\frac{s_1-1}{s_1+1}}(k-m)+\sqrt{\frac{s_2-1}{s_2+1}}(l-k+m)] \\
& [\;\sqrt{\frac{s_1-1}{s_1+1}}\;m+\sqrt{\frac{s_2-1}{s_2+1}}(n-l-m)]A_{k-m}^l A_m^{n-l} \\
& +\;\sum_{l=1}^{n-1}\sum_{m=0}^{n-l}[\;\sqrt{\frac{s_1-1}{s_1+1}}(k-m)+\sqrt{\frac{s_2-1}{s_2+1}}(l-k+m)] \\
& [\;s_1\sqrt{\frac{s_1-1}{s_1+1}}\;m+s_2\sqrt{\frac{s_2-1}{s_2+1}}(n-l-m)]A_{k-m}^l A_m^{n-l}\},
\end{aligned}$$

$n \geq 2,\; 0 \leq k \leq n$; if $k < 0$ or $k > n$ then $A_k^n = 0$;

$$\begin{aligned}
\lambda_{(k,n-k)} \;&= (s_1-1)\sqrt{\frac{s_1-1}{s_1+1}}k(1-k^2) \\
&+\; (s_2-1)\sqrt{\frac{s_2-1}{s_2+1}}(n-k)(1-(n-k)^2) \\
&-\; \frac{(s_1-1)(2s_1+s_2+3)}{s_1+1}\sqrt{\frac{s_2-1}{s_2+1}}k^2(n-k) \\
&-\; \frac{(s_2-1)(s_1+2s_2+3)}{s_2+1}\sqrt{\frac{s_1-1}{s_1+1}}k(n-k)^2.
\end{aligned}$$

Here either $A_0^1 = A_1^1 = 1$ or $A_0^1 = 0, A_1^1 = 1$. If $s_1 < -1,\ s_1 > 1$ and $s_2 < -1,\ s_2 > 1$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $k, n \in Z_+,\ n \geq 2,\ 0 \leq k \leq n$.
Thus, we obtain a 2-soliton solution of the third shallow water waves equation in the form:

$$u(t,x) = \sum_{n=1}^{\infty}\sum_{k=0}^{n}(\sqrt{\frac{s_1-1}{s_1+1}}k+\sqrt{\frac{s_2-1}{s_2+1}}(n-k))A_k^n w_1^k w_2^{n-k}.$$

If $A_0^1 = 0$, then by (29) we obtain

$$v = \sum_{n=1}^{\infty}(-\frac{1}{2})^{n-1}(\sqrt{\frac{s_1+1}{s_1-1}})^{n-1}(\varepsilon w_1)^n = \frac{\varepsilon w_1}{1+\frac{1}{2}\sqrt{\frac{s_1+1}{s_1-1}}\varepsilon w_1} \tag{5.6}$$
$$= \sqrt{\frac{s_1-1}{s_1+1}}\frac{2w}{1+w},$$

where $w = \frac{\varepsilon}{2}\sqrt{\frac{s_1+1}{s_1-1}}w_1$.
In (t, x)-variables we have:

$$v(t,x) = \sqrt{\frac{s_1-1}{s_1+1}}(1 + tanh[\frac{1}{2}\sqrt{\frac{s_1-1}{s_1+1}}(x - s_1 t)]),$$

where we assumed $W_1 = \frac{2}{\varepsilon}\sqrt{\frac{s_1-1}{s_1+1}}$.
Therefore, By (10) we obtain exact soliton solution of the third shallow water waves equation in the form

$$u(t,x) = \frac{s-1}{2(s+1)}sech^2[\frac{1}{2}\sqrt{\frac{s-1}{s+1}}(x - st)].$$

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